

# Refined functional relations for the elliptic SOS model

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## Abstract

In this work we refine the method of [1] and obtain a novel kind of functional equation determining the partition function of the elliptic SOS model with domain wall boundaries. This functional relation is provenient from the dynamical Yang-Baxter algebra and its solution is given in terms of multiple contour integrals.

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## 1 Introduction

Face models or Solid-on-Solid (SOS) models of statistical mechanics were introduced by Baxter in the process of solving the eight-vertex model with periodic boundary conditions [2]. The Boltzmann weights of Baxter's eight-vertex model are parameterised by elliptic functions and this feature is intrinsically connected with the requirement that the model statistical weights satisfy the Yang-Baxter equation [3, 4]. The elliptic nature of the eight-vertex model Boltzmann weights is naturally transported to the corresponding SOS model and a new continuous parameter emerges in the course of Baxter's vertex-face transformation [2]. We shall refer to this new parameter as dynamical parameter [2] and its implications for the analytic theory of the eight-vertex model have been discussed in [5]. Besides the emergence of this new parameter, the resulting statistical weights no longer satisfy the standard Yang-Baxter equation but its dynamical version introduced by Felder [6–8] as the quantised form of a modified classical Yang-Baxter equation [9, 10].

In the same fashion as Drinfeld-Jimbo quantum groups [11–14] provide the algebraic structure underlying the solutions of the Yang-Baxter equation, the so called elliptic quantum groups introduced in [6, 7] accomodate the solutions of the dynamical Yang-Baxter equation. In this work we shall restrict ourselves to the SOS model built out of the solution of the dynamical Yang-Baxter equation associated with the elliptic quantum group  $E_{\tau, \gamma}[\mathfrak{sl}_2]$ . As far as the boundary conditions are concerned, we shall consider the case of domain wall boundaries firstly introduced by Korepin in the context of vertex models [15] and subsequently extended for SOS models in [1, 16–18].

In contrast to the case with periodic boundary conditions, the partition function of vertex and SOS models with domain wall boundaries can be exactly computed without

relying on solutions of Bethe ansatz equations. Interestingly enough, the exact solution of the six-vertex model with domain wall boundaries [19] revealed that the model free-energy differs from the case with periodic boundary conditions [20]. This unusual dependence of bulk thermodynamic properties with boundary conditions has also been observed for the elliptic SOS model when the anisotropy parameter assumes a particular value [21]. For general values of the anisotropy parameter this study still poses as an open problem, probably due to the lack of suitable expressions for the partition function allowing to compute physical properties in the thermodynamic limit. In searching for alternative representations for this partition function, which might render the analysis of the thermodynamic limit feasible, we have obtained in [1] a multiple integral formula for the partition function of the trigonometric SOS model with domain wall boundaries. This case consists of a particular limit of a more general elliptic model, the limit where elliptic theta-functions degenerate into trigonometric functions, and here we refine and generalise the method of [1] for the general elliptic case.

This paper is planned as follows. In the Section 2 we give a brief description of SOS models with domain wall boundaries in terms of the generators of Felder's dynamical Yang-Baxter algebra. The conventions employed here are basically the ones already discussed in [1]. In the Section 3 we demonstrate how the dynamical Yang-Baxter algebra can be explored in order to obtain a functional equation determining the model partition function. This functional equation is solved in Section 4 and concluding remarks are discussed in Section 5. Technical details required throughout this paper are presented in the Appendices.

## 2 Operatorial description of the SOS model

Partition functions of two-dimensional lattice models can be described in terms of operators representing the allowed configurations of the lattice. This feature goes back to Kramers and Wannier transfer matrix technique [22,23] and it has found several important generalisations [24]. Remarkably, when the statistical weights of the model satisfy the Yang-Baxter equation [3] or its dynamical counterpart [6–8], we not only have an operatorial description of the model but also an algebra governing its operators for any size of the lattice. In what follows we shall recall the conventions discussed in [1] which consist of an extension of the ones given in [15] for the six-vertex model.

**Dynamical Yang-Baxter equation.** Following [7,8] we encode the statistical weights of our elliptic SOS model on a matrix  $\mathcal{R} \in \text{End}(\mathbb{V} \otimes \mathbb{V})$  with  $\mathbb{V} \cong \mathbb{C}^2$ . For variables  $\lambda_i, \gamma, \theta \in \mathbb{C}$ , this matrix  $\mathcal{R}$  satisfies the dynamical Yang-Baxter equation,

$$\begin{aligned} \mathcal{R}_{12}(\lambda_1 - \lambda_2, \theta - \gamma \hat{h}_3) \mathcal{R}_{13}(\lambda_1 - \lambda_3, \theta) \mathcal{R}_{23}(\lambda_2 - \lambda_3, \theta - \gamma \hat{h}_1) = \\ \mathcal{R}_{23}(\lambda_2 - \lambda_3, \theta) \mathcal{R}_{13}(\lambda_1 - \lambda_3, \theta - \gamma \hat{h}_2) \mathcal{R}_{12}(\lambda_1 - \lambda_2, \theta) , \end{aligned} \quad (2.1)$$

where  $\hat{h} = \text{diag}(1, -1)$ . The Eq. (2.1) is defined in  $\text{End}(\mathbb{V}_1 \otimes \mathbb{V}_2 \otimes \mathbb{V}_3)$  and the action of  $\mathcal{R}_{12}(\lambda, \theta - \gamma \hat{h}_3)$  on the basis vector  $v_1 \otimes v_2 \otimes v_3$  is understood as

$$[\mathcal{R}(\lambda, \theta - \gamma h) v_1 \otimes v_2] \otimes v_3 , \quad (2.2)$$

where  $h$  is a scalar denoting a particular eigenvalue of  $\hat{h}$ , i.e.  $\hat{h}_i v_i = h v_i$ .

**Definition.** Let  $\tau$  be a complex number such that  $\text{Im}(\tau) > 0$  and write  $p = e^{i\pi\tau}$  so that  $|p| < 1$ . For  $\lambda \in \mathbb{C}$  we define the elliptic function  $f$  with nome  $p$  as

$$f(\lambda) = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^{n-\frac{1}{2}} p^{(n+\frac{1}{2})^2} e^{-(2n+1)\lambda} . \quad (2.3)$$

The function  $f(\lambda)$  corresponds to the Jacobi theta-function  $\Theta_1(i\lambda, \tau)/2$  [25].

The equation (2.1) has been considered in [6, 7] and its solution reads

$$\mathcal{R}(\lambda, \theta) = \begin{pmatrix} a_+(\lambda, \theta) & 0 & 0 & 0 \\ 0 & b_+(\lambda, \theta) & c_+(\lambda, \theta) & 0 \\ 0 & c_-(\lambda, \theta) & b_-(\lambda, \theta) & 0 \\ 0 & 0 & 0 & a_-(\lambda, \theta) \end{pmatrix} \quad (2.4)$$

with non-null entries

$$\begin{aligned} a_{\pm}(\lambda, \theta) &= f(\lambda + \gamma) \\ b_{\pm}(\lambda, \theta) &= f(\lambda) \frac{f(\theta \mp \gamma)}{f(\theta)} \\ c_{\pm}(\lambda, \theta) &= f(\gamma) \frac{f(\theta \mp \lambda)}{f(\theta)} . \end{aligned} \quad (2.5)$$

The algebraic structure underlying (2.4) is the elliptic quantum group  $E_{\tau, \gamma}[\mathfrak{sl}_2]$  and we have collected the properties of  $f$  required through this work in the Appendix B.

**Dynamical monodromy matrix.** Let  $\hat{\theta}_i$  be the operator valued parameter

$$\hat{\theta}_i = \theta - \gamma \sum_{k=i+1}^L \hat{h}_k \quad (2.6)$$

and consider the following ordered product of dynamical  $\mathcal{R}$ -matrices,

$$\mathcal{T}_a(\lambda, \theta) = \prod_{i=1}^L \mathcal{R}_{ai}(\lambda - \mu_i, \hat{\theta}_i) , \quad (2.7)$$

living in the tensor product space  $\mathbb{V}_a \otimes \mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L$ . We shall refer to  $\mathcal{T}_a(\lambda, \theta)$  as dynamical monodromy matrix or simply monodromy matrix. Since the dynamical  $\mathcal{R}$ -matrix (2.4) satisfy the weight-zero condition  $[\mathcal{R}_{ab}(\lambda, \theta), \hat{h}_a + \hat{h}_b] = 0$ , one can show that (2.7) obeys the relation

$$\mathcal{R}_{ab}(\lambda_1 - \lambda_2, \theta - \gamma H) \mathcal{T}_a(\lambda_1, \theta) \mathcal{T}_b(\lambda_2, \theta - \gamma \hat{h}_a) = \mathcal{T}_b(\lambda_2, \theta) \mathcal{T}_a(\lambda_1, \theta - \gamma \hat{h}_b) \mathcal{R}_{ab}(\lambda_1 - \lambda_2, \theta) \quad (2.8)$$

with  $H = \sum_{k=1}^L \hat{h}_k$ . The relation (2.8) shall be referred to as dynamical Yang-Baxter algebra and since  $\mathbb{V} \cong \mathbb{C}^2$ , the dynamical monodromy matrix can be recasted in the form

$$\mathcal{T}_a(\lambda, \theta) = \begin{pmatrix} A(\lambda, \theta) & B(\lambda, \theta) \\ C(\lambda, \theta) & D(\lambda, \theta) \end{pmatrix} \quad (2.9)$$

whose entries are then defined on  $\mathbb{V}_1 \otimes \cdots \otimes \mathbb{V}_L$ .

**Domain wall boundaries.** The partition function of the elliptic SOS model with domain wall boundaries can be written in terms of entries of (2.9) as described in [1]. More precisely, the elliptic SOS model partition function  $Z_\theta$  is given by the expected value

$$Z_\theta = \langle \bar{0} | \prod_{j=1}^L B(\lambda_j, \theta + j\gamma) | 0 \rangle \quad (2.10)$$

where

$$|0\rangle = \bigotimes_{i=1}^L \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad |\bar{0}\rangle = \bigotimes_{i=1}^L \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (2.11)$$

In the next section we shall demonstrate how the dynamical Yang-Baxter algebra can be employed to produce a functional equation determining  $Z_\theta$ .

### 3 Functional relations

The relation (2.8) encodes commutation rules for the operators  $A(\lambda, \theta)$ ,  $B(\lambda, \theta)$ ,  $C(\lambda, \theta)$  and  $D(\lambda, \theta)$  once the structure (2.9) is considered. Out of the sixteen relations contained in (2.8), we will make use of only two of them in order to derive a functional equation describing the partition function (2.10). More precisely, the required relations are simply:

$$\begin{aligned} B(\lambda_1, \theta)B(\lambda_2, \theta + \gamma) &= B(\lambda_2, \theta)B(\lambda_1, \theta + \gamma) \\ A(\lambda_1, \theta + \gamma)B(\lambda_2, \theta) &= \frac{f(\lambda_2 - \lambda_1 + \gamma)}{f(\lambda_2 - \lambda_1)} \frac{f(\theta + \gamma)}{f(\theta + 2\gamma)} B(\lambda_2, \theta + \gamma)A(\lambda_1, \theta + 2\gamma) \\ &\quad - \frac{f(\theta + \gamma - \lambda_2 + \lambda_1)}{f(\lambda_2 - \lambda_1)} \frac{f(\gamma)}{f(\theta + 2\gamma)} B(\lambda_1, \theta + \gamma)A(\lambda_2, \theta + 2\gamma). \end{aligned} \quad (3.1)$$

In addition to that, the weight-zero condition satisfied by (2.4) associated with the definition (2.7) allow us to compute the action of  $A(\lambda, \theta)$  on the states  $|0\rangle$  and  $|\bar{0}\rangle$  defined in (2.11). The vectors  $|0\rangle$  and  $|\bar{0}\rangle$  are respectively the  $\mathfrak{sl}_2$  highest and lowest weight states

and from (2.4) and (2.7) we readily obtain

$$\begin{aligned} A(\lambda, \theta) |0\rangle &= \prod_{j=1}^L f(\lambda - \mu_j + \gamma) |0\rangle \\ \langle \bar{0} | A(\lambda, \theta) &= \frac{f(\theta - \gamma)}{f(\theta + (L-1)\gamma)} \prod_{j=1}^L f(\lambda - \mu_j) \langle \bar{0} | . \end{aligned} \quad (3.2)$$

**The framework.** In order to explore the relations (3.1) and (3.2) we shall consider the quantity

$$\langle \bar{0} | A(\lambda_0, \theta + \gamma) \prod_{j=1}^L B(\lambda_j, \theta + (j-1)\gamma) |0\rangle \quad (3.3)$$

computed in two different ways. We first compute (3.3) using (3.2) to find that it is proportional to  $Z_{\theta-\gamma}(\lambda_1, \dots, \lambda_L)$ . Next we compute (3.3) using the relations (3.1) to move the operator  $A(\lambda_0, \theta + \gamma)$  through the string of operators  $B(\lambda_j, \theta + (j-1)\gamma)$  and then consider its action on the vector  $|0\rangle$ . By doing so we obtain a combination of terms  $Z_\theta$  depending on the set of  $L+1$  variables  $\{\lambda_0, \lambda_1, \dots, \lambda_L\}$  taken  $L$  at a time.

**Functional equation.** The relations (3.2) follows from the  $\mathfrak{sl}_2$  highest weight representation theory and its consistency with the dynamical Yang-Baxter algebra implies the functional equation

$$M_0 Z_{\theta-\gamma}(\lambda_1, \dots, \lambda_L) + \sum_{i=0}^L N_i Z_\theta(\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_L) = 0 \quad (3.4)$$

with coefficients given by

$$\begin{aligned} M_0 &= \frac{f(\theta)}{f(\theta + L\gamma)} \prod_{j=1}^L f(\lambda_0 - \mu_j) \\ N_0 &= -\frac{f(\theta + \gamma)}{f(\theta + (L+1)\gamma)} \prod_{j=1}^L f(\lambda_0 - \mu_j + \gamma) \prod_{j=1}^L \frac{f(\lambda_j - \lambda_0 + \gamma)}{f(\lambda_j - \lambda_0)} \\ N_i &= \frac{f(\theta + \gamma + \lambda_0 - \lambda_i)}{f(\theta + (L+1)\gamma)} \frac{f(\gamma)}{f(\lambda_i - \lambda_0)} \prod_{j=1}^L f(\lambda_i - \mu_j + \gamma) \prod_{\substack{j=1 \\ \neq i}}^L \frac{f(\lambda_j - \lambda_i + \gamma)}{f(\lambda_j - \lambda_i)} \\ &\hspace{25em} i = 1, \dots, L \end{aligned} \quad (3.5)$$

Some remarks are in order at this stage. Although the partition function considered here reduces to the one studied in [1] when the elliptic theta-function  $f$  degenerate into a trigonometric function, the functional equation (3.4) still differs significantly from the one obtained in [1]. For instance, (3.4) is a functional equation also over the variable  $\theta$

and even in the limit  $\theta \rightarrow \infty$ , where  $Z_{\theta-\gamma}$  and  $Z_\theta$  coincide, we still would be left with a functional equation different from the one presented in [26]. This divergence is due to the fact that here we have started our analysis with the quantity (3.3) instead of  $\langle \bar{0} | C(\lambda_0, \theta + \gamma) \prod_{j=1}^{L+1} B(\lambda_j, \theta + (j-1)\gamma) | 0 \rangle$  as employed in the works [1] and [26]. This different starting point allows us to obtain a simpler functional equation whose solution will be discussed in the next section.

## 4 The partition function

This section is concerned with solving the functional relation (3.4). The method we shall employ is essentially the one described in [1] which exploits special zeroes of  $Z_\theta$  to produce a separation of variables. Some structural properties of (3.4) will be of utility to help us identifying the elements required to solve this functional equation. For instance, the partition function  $Z_\theta$  is a function of two sets of variables, i.e.  $\{\lambda_1, \dots, \lambda_L\}$  and  $\{\mu_1, \dots, \mu_L\}$ , in addition to the parameters  $\gamma$ ,  $\theta$  and the elliptic nome  $p$ . In our framework, however, the set of variables  $\{\mu_1, \dots, \mu_L\}$  can also be regarded as parameters while  $\theta$  is promoted to a variable. This follows from the fact that (3.4) is an equation not only over variables  $\lambda_j$  but also  $\theta$ .

With this in mind we can see that (3.4) is a homogeneous equation in the sense that  $\alpha Z_\theta$  is a solution if so is  $Z_\theta$  and  $\alpha$  is independent of  $\lambda_j$  and  $\theta$ . This property implies that the equation (3.4) will be able to determine the partition function up to an overall multiplicative factor independent of  $\lambda_j$  and  $\theta$  at most. Thus the complete determination of  $Z_\theta$  will require that we are able to compute it for a particular value of  $\lambda_j$  and  $\theta$  in order to determine this overall factor. Any point on the  $(\lambda_j, \theta)$ -plane would serve our need and we can choose it such that the evaluation of  $Z_\theta$  is as simple as possible. As demonstrated in the Appendix A, the evaluation of  $Z_\theta$  in the limit  $(\lambda_j, \theta) \rightarrow \infty$  can be performed in the same lines of [26]. Moreover, the equation (3.4) is linear which raises the issue of uniqueness of the solution since linear combinations of particular solutions also solves (3.4). Similarly to the case considered in [1], we will see that the location of zeroes of  $Z_\theta$  will select the appropriate solution uniquely. These properties are discussed in the Appendices A and B, and they are summarised as follows.

**Asymptotic behaviour.** In the limit  $(\lambda_j, \theta) \rightarrow \infty$  the partition function (2.10) behaves as

$$\begin{aligned}
Z_\theta(\lambda_1, \dots, \lambda_L) \sim & \frac{f(\gamma)^L}{2^{L(L-1)}} \sum_{n_1^{(1)}=-\infty}^{\infty} \dots \sum_{n_{L-1}^{(1)}=-\infty}^{\infty} \dots \sum_{n_1^{(L)}=-\infty}^{\infty} \dots \sum_{n_{L-1}^{(L)}=-\infty}^{\infty} (-1)^{\sum_{a=1}^L \sum_{i=1}^{L-1} n_i^{(a)} - \frac{L(L-1)}{2}} \\
& \prod_{a=1}^L \prod_{i=1}^{L-1} p_{n_i^{(a)}} q_{n_i^{(a)}} e_{n_i^{(a)}}^{\lambda_a - \mu_i^{(a)}} \sum_{\sigma \in \mathcal{S}_L} \prod_{(a,b) \in I_\sigma} (q_{n_{b-1}^{(a)}} q_{n_a^{(b)}})^{-1}, \quad (4.1)
\end{aligned}$$

where  $e_n = e^{-(2n+1)}$ ,  $p_n = p^{(n+\frac{1}{2})^2}$ ,  $q_n = e_n^\gamma$  and  $\mu^{(a)} = \{\mu_i : i \neq a\}$ . Here  $\mathcal{S}_L$  denotes the group of permutations of  $L$  objects and  $\sigma = \sigma(1) \dots \sigma(L)$  stands for a given permutation.

The set of inversion vertices for a given  $\sigma$  is denoted by  $I_\sigma$ .

**Higher order theta-function.** The partition function  $Z_\theta$  is a theta function of order  $L$  and norm  $t_i$  in each one of its variables  $\lambda_i$  separately. That means it can be factorised as

$$Z_\theta = C \prod_{j=1}^L f(\lambda_i - \xi_j^{(i)}) \quad (4.2)$$

where  $C$  is  $\lambda_i$  independent and the zeroes  $\xi_j^{(i)}$  satisfy  $\xi_1^{(i)} + \dots + \xi_L^{(i)} = t_i$ . Unveiling special zeroes  $\xi_j^{(i)}$  will be an important step for solving (3.4).

Now we proceed with the analysis of (3.4) in the lines of [1]. For that we look for special values of the variables  $\lambda_j$  such that particular zeroes of  $Z_\theta$  can be identified.

**Special zeroes.** The coefficients  $M_0$  and  $N_i$  given in (3.5) exhibit a factorised form and due to that identifying their zeroes is a simple task. For instance, when  $\lambda_0 = \mu_1$  and  $\lambda_1 = \mu_1 - \gamma$  we find that  $M_0 = N_0 = N_1 = 0$ . Next we can set  $\lambda_j = \lambda_{j+1} - \gamma$  successively for  $j \in [2, L-1]$  and collect the result at each step. At the last step we find

$$N_2 Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_L - (L-2)\gamma, \lambda_L - (L-3)\gamma, \dots, \lambda_L) = 0, \quad (4.3)$$

and since  $N_2$  is finite we can conclude that the vanishing of (4.3) is due to  $Z_\theta$ . This result can now be substituted back into the previous steps leading to (4.3). By doing so we find the more general vanishing condition, namely  $Z_\theta(\mu_1, \mu_1 - \gamma, \lambda_3, \dots, \lambda_L) = 0$ , for general values of the variables  $\lambda_j$  with  $j \in [3, L]$ . Now this process can be repeated starting with variables  $\lambda_0 = \mu_1$  and  $\lambda_j = \mu_1 - \gamma$  for any  $j \in [1, L]$ , which allows us to conclude that  $Z_\theta(\mu_1, \dots, \mu_1 - \gamma, \dots) = 0$ .

**Building up the solution.** The zeroes of  $Z_\theta$  above unveiled have a special appeal since we are interested in the solution of (3.4) consisting of a higher order theta-function (4.2). Taking that into account, those special zeroes imply that

$$Z_\theta(\mu_1, \lambda_2, \dots, \lambda_L) = \prod_{j=2}^L f(\lambda_j - \mu_1 + \gamma) V_\theta(\lambda_2, \dots, \lambda_L), \quad (4.4)$$

where  $V_\theta$  is also a theta-function but of order  $L-1$  in each one of its variables. Next we substitute the expression (4.4) back into the equation (3.4) and set  $\lambda_0 = \mu_1$ . The resulting equation can then be solved for  $Z_\theta(\lambda_1, \dots, \lambda_L)$  yielding the expression

$$Z_\theta(\lambda_1, \dots, \lambda_L) = \sum_{i=1}^L m_i V_\theta(\dots, \lambda_{i-1}, \lambda_{i+1}, \dots) \quad (4.5)$$



with coefficients

$$m_i = \frac{f(\theta + \gamma + \mu_1 - \lambda_i)}{f(\theta + \gamma)} \prod_{j=2}^L \frac{f(\lambda_i - \mu_j + \gamma)}{f(\mu_1 - \mu_j + \gamma)} \prod_{\substack{j=1 \\ j \neq i}}^L f(\lambda_j - \mu_1) \frac{f(\lambda_j - \lambda_i + \gamma)}{f(\lambda_j - \lambda_i)}. \quad (4.6)$$

We then substitute the formula (4.5) back into the original equation (3.4) and set  $\lambda_L = \mu_1$ . After eliminating an overall factor we are left with the equation

$$P_0 V_{\theta-\gamma}(\lambda_1, \dots, \lambda_{L-1}) + \sum_{i=0}^{L-1} Q_i V_{\theta}(\lambda_0, \dots, \lambda_{i-1}, \lambda_{i+1}, \dots, \lambda_{L-1}) = 0, \quad (4.7)$$

where the coefficients  $P_0$  and  $Q_i$  correspond respectively to the coefficients  $M_0$  and  $N_i$  given in (3.5) under the mapping  $L \rightarrow L - 1$ ,  $\theta \rightarrow \theta + \gamma$  and  $\mu_i \rightarrow \mu_{i+1}$ . Thus the function  $V_{\theta}$  obeys essentially the same equation as the partition function  $Z_{\theta}$  but for a square lattice of dimensions  $(L - 1) \times (L - 1)$ . Now since  $V_{\theta}$  is also a theta-function, this procedure can be repeatedly carried out until we reach the equation for  $L = 1$ . The solution of (3.4) for  $L = 1$  can be found in the Appendix D and gathering our results we obtain the following solution for general  $L$ ,

$$Z_{\theta}(\lambda_1, \dots, \lambda_L) = \sum_{\sigma \in \mathcal{S}_L} F_{\sigma(1) \dots \sigma(L)} \quad (4.8)$$

where

$$\begin{aligned} F_{\sigma(1) \dots \sigma(L)} = & \frac{\Omega_L}{\prod_{k=2}^L f(\mu_1 - \mu_k + \gamma)} \prod_{n=1}^L \frac{f(\theta + n\gamma - \lambda_{\sigma(n)} + \mu_n)}{f(\theta + n\gamma)} \prod_{j>n}^L f(\lambda_{\sigma(n)} - \mu_j + \gamma) \prod_{j<n}^L f(\lambda_{\sigma(n)} - \mu_j) \\ & \times \prod_{m>n}^L \frac{f(\lambda_{\sigma(m)} - \lambda_{\sigma(n)} + \gamma)}{f(\lambda_{\sigma(m)} - \lambda_{\sigma(n)})}. \end{aligned} \quad (4.9)$$

The overall factor  $\Omega_L$  arises from the homogeneity of (3.4) as previously discussed, and from (4.1) we obtain  $\Omega_L = f(\gamma)^L \prod_{k=2}^L f(\mu_1 - \mu_k + \gamma)$ . It is important to remark here that this partition function has also been considered in [16–18] where a similar but still different expression for  $F_{\sigma(1) \dots \sigma(L)}$  has been found.

**Multiple integral formula.** The partition function  $Z_{\theta}$  can be represented by a multiple contour integral as follows. The function  $V_{\theta}$  in the formula (4.5) is essentially our partition function for a lattice of size  $(L - 1) \times (L - 1)$  and modified parameters. In fact, the decomposition of  $Z_{\theta}$  in terms of  $V_{\theta}$  as described by (4.5) can be thought of as a separation of variables. Moreover, the prescription given by (4.5) can be mimicked by the Cauchy like integral

$$Z_{\theta}(\lambda_1, \dots, \lambda_L) = \oint \dots \oint \frac{H_L(\{\lambda_j\}|\{w_j\})}{\prod_{i,j=1}^L f(w_i - \lambda_j)} \prod_{j=1}^L \frac{dw_j}{2i\pi} \quad (4.10)$$

with integration contours enclosing solely the zeroes of  $f$  when  $w_i \rightarrow \lambda_j$ . Also we assume that  $H_L(\{\lambda_j\}|\{w_j\})$  has no poles inside the integration contour.

For the case  $L = 1$ , we can immediately read from (D.6) that

$$H_1(\lambda_1|w_1) = f'(0)f(\gamma)\frac{f(\theta + \gamma - w_1 + \mu_1)}{f(\theta + \gamma)} \quad (4.11)$$

where  $f'(0)$  denotes the derivative of  $f(\lambda)$  with respect to  $\lambda$  at the point  $\lambda = 0$ . Guided by this result when looking to (4.5) we then find

$$\begin{aligned} H_L(\{\lambda_j\}|\{w_j\}) = & [f'(0)f(\gamma)]^L \prod_{j>i}^L f(w_j - w_i + \gamma)f(w_j - w_i) \prod_{j=1}^L \frac{f(\theta + j\gamma - w_j + \mu_j)}{f(\theta + j\gamma)} \times \\ & \prod_{j<i}^L f(\mu_j - w_i) \prod_{j>i}^L f(w_i - \mu_j + \gamma) . \end{aligned} \quad (4.12)$$

The expression (4.12) already takes into account the asymptotic behaviour (4.1) and though here we have considered a functional equation different from the one obtained in [1], the expression (4.12) indeed reduces to the formula of [1] in the degenerated limit. Moreover, it is worth remarking that the homogeneous limit  $\lambda_j \rightarrow \lambda$  and  $\mu_j \rightarrow \mu$  can be trivially obtained from the integral formula (4.10, 4.12).

## 5 Concluding remarks

In this work the partition function of the elliptic SOS model with domain wall boundaries was studied through a fusion of algebraic and functional techniques. The partition function of the model was shown to obey a functional equation provenient from the dynamical Yang-Baxter algebra which is valid for general values of the model parameters. The solution was then obtained as a multiple contour integral.

The possibility of deriving functional equations for such partition functions from the Yang-Baxter algebra and its dynamical counterpart was firstly demonstrated in [26, 1]. Although here we have also employed the dynamical Yang-Baxter algebra, the mechanism considered in Section 3 differs from the one used in [26, 1], and the resulting functional equation is significantly simpler than the ones previously obtained. Interestingly, solving this new type of functional equation follows the same lines of [1] but each one of the steps required are dramatically simplified.

The elliptic SOS model considered here is also referred to as 8VSOS model in the literature and for the special value of the anisotropy parameter  $\gamma = \frac{2i\pi}{3}$ , it reduces to the so called Three-colouring model [27]. For the case with domain wall boundaries, the partition function of the Three-colouring model was shown to obey a certain functional equation in [28, 29] but a possible connection with our results has eluded us so far. In the work [21] this same partition function was studied under the light of the symmetric polynomials theory where a set of two-variables polynomials have been introduced. These polynomials were conjectured in [30] to satisfy a certain partial differential equation and recurrence relation, to which (3.4), (4.10) and (4.12) might shed some light into their proofs.

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## A Asymptotic behaviour

In the limit  $\theta \rightarrow \infty$  the  $\mathcal{R}$ -matrix (2.4) resembles the one associated with the six-vertex model, except that the Boltzmann weights (2.5) still consist of elliptic theta-functions. Moreover, from the definition (2.10) we can readily see that the whole dependence of  $Z_\theta$  with a particular variable  $\lambda_j$  will be described by the operator  $B(\lambda_j, \theta + j\gamma)$ , which is very similar to the six-vertex model analogous in the proposed limit. Thus considering the definition (2.3), in the limit  $(\lambda_j, \theta) \rightarrow \infty$  we obtain

$$B(\lambda_j, \theta + j\gamma) \sim \frac{f(\gamma)}{2^{L-1}} \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_{L-1}=-\infty}^{\infty} (-1)^{\sum_{i=1}^{L-1} n_i - \frac{(L-1)}{2}} \prod_{i=1}^{L-1} p_{n_i} q_{n_i}^{\frac{1}{2}} e_{n_i}^{\lambda_j} \times \sum_{j=1}^L e_{n_1}^{-\mu_1} \cdots e_{n_{j-1}}^{-\mu_{j-1}} P_j^{\vec{n}} e_{n_j}^{-\mu_{j+1}} \cdots e_{n_{L-1}}^{-\mu_L} . \quad (\text{A.1})$$

In the expression (A.1) we have introduced the conventions  $e_n = e^{-(2n+1)}$ ,  $p_n = p^{(n+\frac{1}{2})^2}$ ,  $q_n = e_n^\gamma$  and  $\vec{n} = (n_1, \dots, n_{L-1})$ , while the operator  $P_j^{\vec{n}}$  reads

$$P_j^{\vec{n}} = K_{n_1} \otimes \cdots \otimes K_{n_{j-1}} \otimes X^- \otimes K_{n_{j+1}}^{-1} \otimes \cdots \otimes K_{n_{L-1}}^{-1} . \quad (\text{A.2})$$

In their turn the operators  $K_n$  and  $X^-$  are defined as

$$K_n = \begin{pmatrix} q_n^{\frac{1}{2}} & 0 \\ 0 & q_n^{-\frac{1}{2}} \end{pmatrix} \quad X^- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (\text{A.3})$$

and, together with  $X^+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ , they satisfy the following analogous of the  $q$ -deformed  $\mathfrak{su}_2$  algebra

$$\begin{aligned} K_n X^\pm K_m^{-1} &= q_{\frac{n+m}{2}}^\pm X^\pm \\ [X^+, X^-] &= \frac{K_n K_m - K_n^{-1} K_m^{-1}}{(q_{\frac{n+m}{2}} - q_{\frac{n+m}{2}}^{-1})} . \end{aligned} \quad (\text{A.4})$$

The relations (A.4) allow us to demonstrate the properties

$$\begin{aligned} P_i^{\vec{n}^{(a)}} P_j^{\vec{n}^{(b)}} &= q_{n_i^{(b)}} q_{n_j^{(a)}} P_j^{\vec{n}^{(b)}} P_i^{\vec{n}^{(a)}} \quad (i < j) \\ P_i^{\vec{n}^{(a)}} P_i^{\vec{n}^{(b)}} &= 0 \end{aligned} \quad (\text{A.5})$$

which will be of utility for the analysis of the asymptotic behaviour of  $Z_\theta$ .

In order to analyse the behaviour of  $Z_\theta$  in the proposed limit we now substitute the expansion (A.1) into the definition (2.10), and use the relations (A.5) to reorganise the result properly. By doing so we obtain

$$\begin{aligned}
Z_\theta(\lambda_1, \dots, \lambda_L) \sim & \frac{f(\gamma)^L}{2^{L(L-1)}} \sum_{n_1^{(1)}=-\infty}^{\infty} \cdots \sum_{n_{L-1}^{(1)}=-\infty}^{\infty} \cdots \sum_{n_1^{(L)}=-\infty}^{\infty} \cdots \sum_{n_{L-1}^{(L)}=-\infty}^{\infty} (-1)^{\sum_{a=1}^L \sum_{i=1}^{L-1} n_i^{(a)} - \frac{L(L-1)}{2}} \\
& \prod_{a=1}^L \prod_{i=1}^{L-1} p_{n_i^{(a)}} q_{n_i^{(a)}}^{\frac{1}{2}} e^{\frac{\lambda_a - \mu_i^{(a)}}{n_i^{(a)}}} \sum_{\sigma \in \mathcal{S}_L} \prod_{(a,b) \in I_\sigma} (q_{n_{b-1}^{(a)}} q_{n_a^{(b)}})^{-1} \langle \bar{0} | \prod_{a=1}^L P_a^{\vec{n}^{(a)}} | 0 \rangle
\end{aligned} \tag{A.6}$$

where  $\mu^{(a)} = \{\mu_i : i \neq a\}$ . As usual  $\mathcal{S}_L$  denotes the group of permutations of  $L$  objects and  $\sigma(a)$  stands for the permutation of the  $a$ -th object. In order to clarify the meaning of  $I_\sigma$  let us consider the usual two row representation of  $\sigma$ . We draw a line starting at the object  $a$  in the top row and ending in the bottom row at the position  $\sigma(a)$  such that only two lines intersect at any one point. The lines are labelled by their numbers in the top row and the points of intersection are called inversion vertices. In this way an inversion vertex can be labelled by a pair  $(a, b)$  with  $a < b$  such that  $a$  and  $b$  label the two intersecting lines originating the inversion vertex. Then denoting  $[L] = \{1, \dots, L\}$ , we call  $I_\sigma = \{(a, b) \in [L] \times [L] : a < b \text{ and } \sigma(a) > \sigma(b)\}$  the set of inversion vertices labels of a given permutation  $\sigma$ .

The next step to obtain an explicit expression for (A.6) is to compute the quantity  $\langle \bar{0} | \prod_{a=1}^L P_a^{\vec{n}^{(a)}} | 0 \rangle$  which can be readily performed since the operators  $P_a^{\vec{n}^{(a)}}$  consist of a simple tensor product (A.2). Thus considering (2.11) we obtain

$$\langle \bar{0} | \prod_{a=1}^L P_a^{\vec{n}^{(a)}} | 0 \rangle = \prod_{a=1}^L \prod_{i=1}^{L-1} q_{n_i^{(a)}}^{\frac{1}{2}} \tag{A.7}$$

which can be substituted in (A.6) yielding

$$\begin{aligned}
Z_\theta(\lambda_1, \dots, \lambda_L) \sim & \frac{f(\gamma)^L}{2^{L(L-1)}} \sum_{n_1^{(1)}=-\infty}^{\infty} \cdots \sum_{n_{L-1}^{(1)}=-\infty}^{\infty} \cdots \sum_{n_1^{(L)}=-\infty}^{\infty} \cdots \sum_{n_{L-1}^{(L)}=-\infty}^{\infty} (-1)^{\sum_{a=1}^L \sum_{i=1}^{L-1} n_i^{(a)} - \frac{L(L-1)}{2}} \\
& \prod_{a=1}^L \prod_{i=1}^{L-1} p_{n_i^{(a)}} q_{n_i^{(a)}} e^{\frac{\lambda_a - \mu_i^{(a)}}{n_i^{(a)}}} \sum_{\sigma \in \mathcal{S}_L} \prod_{(a,b) \in I_\sigma} (q_{n_{b-1}^{(a)}} q_{n_a^{(b)}})^{-1}
\end{aligned} \tag{A.8}$$

in the limit  $(\lambda_j, \theta) \rightarrow \infty$ .

## B Theta-function properties

In this appendix we recall some useful properties of elliptic theta-functions that we have considered through this paper. We remark here that many of these properties have also been discussed in [16]. The function  $f$  defined in Section 2 consists basically of the Jacobi theta-function  $\Theta_1$  [25] and in this paper we have omitted the dependence of  $f$  with the elliptic nome  $p$  for brevity. In what follows we summarise some properties of elliptic theta-functions adjusted to our conventions.

**Addition rule.** The function  $f$  satisfy the addition rule

$$\begin{aligned} f(\lambda_1 + \lambda_2)f(\lambda_1 - \lambda_2)f(\lambda_3 + \lambda_4)f(\lambda_3 - \lambda_4) &= \\ f(\lambda_1 + \lambda_4)f(\lambda_1 - \lambda_4)f(\lambda_3 + \lambda_2)f(\lambda_3 - \lambda_2) &+ f(\lambda_1 + \lambda_3)f(\lambda_1 - \lambda_3)f(\lambda_2 + \lambda_4)f(\lambda_2 - \lambda_4) . \end{aligned} \quad (\text{B.1})$$

**Analiticity and periodicity.** The function  $f$  is an entire function, that is to say all of its singularities are removable, and it has only simple zeroes. It is also an odd function and quasi doubly-periodic, i.e.

$$f(\lambda - i\pi) = -f(\lambda) \quad f(\lambda - i\pi\tau) = -e^{2\lambda - i\pi\tau} f(\lambda) . \quad (\text{B.2})$$

**Trigonometric limit.** In the limit  $p \rightarrow 0$  we observe the degeneracy  $\lim_{p \rightarrow 0} f(\lambda) = \sinh(\lambda)$ . This fact motivates the use of the definition (2.3) which allows for an easy comparison with previous results in the literature.

**Higher order theta-functions.** For a fixed value of the elliptic nome  $\tau$ , we call  $\mathcal{F}$  a theta-function of order  $L$  and norm  $t$  if

$$\mathcal{F}(\lambda) = \prod_{j=1}^L f(\lambda - \chi_j) \quad (\text{B.3})$$

for constants  $\chi_j$  such that  $\sum_{j=1}^L \chi_j = t$ . Moreover, due to (B.2) one can readily show the quasi-periodicity

$$\begin{aligned} \mathcal{F}(\lambda - i\pi) &= (-1)^L \mathcal{F}(\lambda) \\ \mathcal{F}(\lambda - i\pi\tau) &= (-1)^L e^{2(L\lambda - t) - i\pi\tau L} \mathcal{F}(\lambda) . \end{aligned} \quad (\text{B.4})$$

In fact, the factorised form (B.3) and the quasi-periodicity (B.4) for entire functions can be shown to be equivalent properties [31]. This feature allows us to state a more general result. Let  $\bar{\mathcal{F}}$  be defined as

$$\bar{\mathcal{F}}(\lambda) = \sum_i \prod_{j=1}^L f(\lambda - \chi_j^{(i)}) , \quad (\text{B.5})$$

such that  $\sum_{j=1}^L \chi_j^{(i)} = t$  for any  $i$ . The function  $\bar{\mathcal{F}}$  is entire and obeys the quasi-periodicity (B.5), thus it can be factorised similarly to (B.3).

**$Z_\theta$  as a higher order theta-function.** The partition function  $Z_\theta$  defined in (2.10) is written as a product of operators  $B(\lambda, \theta)$ . As a matter of fact, the whole dependence of  $Z_\theta$  with a particular variable  $\lambda_j$  is contained in a single operator  $B(\lambda_j, \theta + j\gamma)$  since the vectors  $|0\rangle$  and  $|\bar{0}\rangle$  are constants. From the definitions (2.4), (2.5), (2.7) and (2.9) we can clearly see that the entries of  $B(\lambda_j, \theta + j\gamma)$  will be of the form (B.5). Thus the partition function  $Z_\theta$  is a theta-function in the variable  $\lambda_j$  of order  $L$  and norm  $t_j$ .

## C $Z_\theta$ as a symmetric function

The commutativity of operators  $B(\lambda, \theta)$  as described by (3.1), together with the definition (2.10), implies that  $Z_\theta$  is a symmetric function. This commutativity has been extensively employed in the derivation of (3.4) and here we intend to show that this symmetry becomes an inherent property of the solutions of (3.4).

Through the inspection of the coefficients (3.5), we notice that  $N_i \leftrightarrow N_j$  under the mapping  $\lambda_i \leftrightarrow \lambda_j$  while  $M_0 \rightarrow M_0$  and  $N_k \rightarrow N_k$  for  $k \neq i, j$ . Thus performing this mapping on (3.4) and subtracting it from the original equation we obtain the following relation,

$$\begin{aligned}
& M_0[Z_{\theta-\gamma}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_L) - Z_{\theta-\gamma}(\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_L)] \\
& + N_0[Z_\theta(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_L) - Z_\theta(\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_L)] \\
& + \sum_{\substack{k=1 \\ \neq i, j}}^L N_k Z_\theta(\lambda_0, \dots, \lambda_i, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_j, \dots) \\
& - \sum_{\substack{k=1 \\ \neq i, j}}^L N_k Z_\theta(\lambda_0, \dots, \lambda_j, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_i, \dots) = 0.
\end{aligned} \tag{C.1}$$

Next we solve (C.1) for the  $l$ -th term of the summation over the index  $k$  which yields the expression

$$\begin{aligned}
& \frac{M_0}{N_l}[Z_{\theta-\gamma}(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_L) - Z_{\theta-\gamma}(\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_L)] \\
& + \frac{N_0}{N_l}[Z_\theta(\lambda_1, \dots, \lambda_i, \dots, \lambda_j, \dots, \lambda_L) - Z_\theta(\lambda_1, \dots, \lambda_j, \dots, \lambda_i, \dots, \lambda_L)] \\
& + \sum_{\substack{k=1 \\ \neq i, j, l}}^L \frac{N_k}{N_l} Z_\theta(\lambda_0, \dots, \lambda_i, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_j, \dots) \\
& - \sum_{\substack{k=1 \\ \neq i, j, l}}^L \frac{N_k}{N_l} Z_\theta(\lambda_0, \dots, \lambda_j, \dots, \lambda_{k-1}, \lambda_{k+1}, \dots, \lambda_i, \dots) = \\
& Z_\theta(\lambda_0, \dots, \lambda_j, \dots, \lambda_{l-1}, \lambda_{l+1}, \dots, \lambda_i, \dots) - Z_\theta(\lambda_0, \dots, \lambda_i, \dots, \lambda_{l-1}, \lambda_{l+1}, \dots, \lambda_j, \dots).
\end{aligned} \tag{C.2}$$

The RHS of (C.2) does not depend on  $\lambda_l$  so this variable can be chosen such that the LHS of (C.2) vanishes. Thus we can conclude that

$$Z_\theta(\lambda_0, \dots, \lambda_j, \dots, \lambda_{l-1}, \lambda_{l+1}, \dots, \lambda_i, \dots) = Z_\theta(\lambda_0, \dots, \lambda_i, \dots, \lambda_{l-1}, \lambda_{l+1}, \dots, \lambda_j, \dots), \quad (\text{C.3})$$

and since this is valid for any  $i, j$  and  $l$ , the symmetry property

$$Z_\theta(\dots, \lambda_i, \dots, \lambda_j, \dots) = Z_\theta(\dots, \lambda_j, \dots, \lambda_i, \dots) \quad (\text{C.4})$$

immediately follows.

## D Solution for $L = 1$

The functional equation (3.4) for  $L = 1$  explicitly reads

$$M_0 Z_{\theta-\gamma}(\lambda_1) + N_0 Z_\theta(\lambda_1) + N_1 Z_\theta(\lambda_0) \quad (\text{D.1})$$

with coefficients

$$\begin{aligned} M_0 &= \frac{f(\theta)}{f(\theta + \gamma)} f(\lambda_0 - \mu_1) \\ N_0 &= -\frac{f(\theta + \gamma)}{f(\theta + 2\gamma)} f(\lambda_0 - \mu_1 + \gamma) \frac{f(\lambda_1 - \lambda_0 + \gamma)}{f(\lambda_1 - \lambda_0)} \\ N_1 &= \frac{f(\theta + \gamma + \lambda_0 - \lambda_1)}{f(\theta + 2\gamma)} \frac{f(\gamma)}{f(\lambda_1 - \lambda_0)} f(\lambda_1 - \mu_1 + \gamma). \end{aligned} \quad (\text{D.2})$$

By setting  $\lambda_0 = \lambda_1 - \theta - \gamma$ , the coefficient  $N_1$  vanishes and (D.1) simplifies to

$$Z_\theta(\lambda_1) \frac{f(\theta + \gamma)}{f(\theta + \gamma + \mu_1 - \lambda_1)} = Z_{\theta-\gamma}(\lambda_1) \frac{f(\theta)}{f(\theta + \mu_1 - \lambda_1)}. \quad (\text{D.3})$$

The relation (D.3) is an equation only over the variable  $\theta$  which is readily solved by

$$Z_\theta(\lambda_1) = \frac{f(\theta + \gamma - \lambda_1 + \mu_1)}{f(\theta + \gamma)} F(\lambda_1) \quad (\text{D.4})$$

where  $F$  is  $\theta$  independent. After eliminating the dependence with  $\theta$ , we can substitute (D.4) back into (D.1). The resulting equation can then be simplified and we obtain the relation

$$\frac{f(\gamma) f(\theta + \gamma + \mu_1 - \lambda_0) f(\theta + \gamma + \lambda_0 - \lambda_1) f(\lambda_1 - \mu_1 + \gamma)}{f(\theta + \gamma) f(\theta + 2\gamma) f(\lambda_1 - \lambda_0)} (F(\lambda_1) - F(\lambda_0)) = 0. \quad (\text{D.5})$$

From (D.5) we can conclude that  $F$  is a constant and it can be fixed by the asymptotic behaviour (A.8). Thus we find for  $L = 1$ ,

$$Z_\theta(\lambda) = f(\gamma) \frac{f(\theta + \gamma - \lambda + \mu_1)}{f(\theta + \gamma)}. \quad (\text{D.6})$$

## References

- [1] W. Galleas. Multiple integral representation for the trigonometric SOS model with domain wall boundaries. *Nucl. Phys. B*, 858(1):117–141, 2012, math-ph/1111.6683.
- [2] R. J. Baxter. Eight-vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain II. Equivalence to a generalized ice-type model. *Ann. Phys.*, 76:25, 1973.
- [3] R. J. Baxter. Eight vertex model in lattice statistics. *Phys. Rev. Lett.*, 26:832, 1971.
- [4] R. J. Baxter. Partition function of 8-vertex lattice model. *Ann. Phys.*, 70:193, 1972.
- [5] V. V. Bazhanov and V. V. Mangazeev. Analytic theory of the eight-vertex model. *Nucl. Phys. B*, 775(3):225–282, 2007.
- [6] G. Felder. Elliptic quantum groups. 1994, hep-th/9412207.
- [7] G. Felder. Conformal field theory and integrable systems associated to elliptic curves. *Proceedings of the International Congress of Mathematicians*, 1:1247, 1995.
- [8] G. Felder. Algebraic bethe ansatz for the elliptic quantum group  $e_{\tau,\eta}(sl_2)$ . *Nucl. Phys. B*, 480:485, 1996.
- [9] D. Bernard. On the Wess-Zumino-Witten model on the torus. *Nucl. Phys. B*, 303:77, 1988.
- [10] D. Bernard. On the Wess-Zumino-Witten model on Riemann surfaces. *Nucl. Phys. B*, 309:145, 1988.
- [11] V. G. Drinfel’d. Hopf algebras and the quantum Yang-Baxter equation. *Sov. Math. Dokl.*, 32:254–258, 1985.
- [12] M. Jimbo. Quantum  $R$ -matrix for the generalized Toda system. *Commun. Math. Phys.*, 102:537–547, 1986.
- [13] M. Jimbo. A  $q$ -analog of  $U(gl(n+1))$ , Hecke Algebra and the Yang-Baxter equation. *Lett. Math. Phys.*, 11:247, 1986.
- [14] M. Jimbo. A  $q$ -difference analog of  $U(g)$  and the Yang-Baxter equation. *Lett. Math. Phys.*, 10:63–69, 1985.
- [15] V. E. Korepin. Calculation of norms of Bethe wave functions. *Commun. Math. Phys.*, 86:391–418, 1982. 10.1007/BF01212176.
- [16] H. Rosengren. An Izergin-Korepin type identity for the 8VSOS model with applications to alternating sign matrices. *Adv. Appl. Math.*, 43:137, 2009.
- [17] S. Pakuliak, V. Rubtsov, and A. Silantyev. SOS model partition function and the elliptic weight function. *J. Phys. A*, 41:295204, 2008.



- [18] W.-L. Yang and Y.-Z. Zhang. Partition function of the eight-vertex model with domain wall boundary condition. *J. Math. Phys.*, 50:083518, 2009.
- [19] A. G. Izergin. Statistical sum of the 6-vertex model in a finite lattice. *Sov. Phys. Dokl.*, 32:878, 1987.
- [20] V. E. Korepin and P. Zinn-Justin. Thermodynamic limit of the six-vertex model with domain wall boundary conditions. *J. Phys. A: Math. Gen.*, 33:7053, 2000.
- [21] H. Rosengren. The three-colour model with domain wall boundary conditions. *Adv. Appl. Math.*, 46:481, 2011.
- [22] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet Part I. *Phys. Rev.*, 60(3):252, 1941.
- [23] H. A. Kramers and G. H. Wannier. Statistics of the two-dimensional ferromagnet Part II. *Phys. Rev.*, 60(3):263, 1941.
- [24] R. J. Baxter. *Exactly Solved Models in Statistical Mechanics*. Dover Publications, Inc., Mineola, New York, 2007.
- [25] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press, fourth edition, 1927.
- [26] W. Galleas. Functional relations for the six-vertex model with domain wall boundary conditions. *J. Stat. Mech.*, 2010(06):P06008, 2010.
- [27] R. J. Baxter. Three-colorings of the square lattice: A hard squares model. *J. Math. Phys.*, 11(10):3116, 1970.
- [28] A. G. Razumov and Y. G. Stroganov. Three-coloring statistical model with domain wall boundary conditions: Functional equations. *Theor. Math. Phys.*, 161:1325, 2009.
- [29] A. G. Razumov and Y. G. Stroganov. Three-coloring statistical model with domain wall boundary conditions: Trigonometric limit. *Theor. Math. Phys.*, 161:1451, 2009.
- [30] V. V. Mangazeev and V. V. Bazhanov. The eight-vertex model and Painleve VI equation II: eigenvector results. *J. Phys. A-Math. Theor.*, 43(8), 2010.
- [31] H. Weber. *Elliptische Functionen und algebraische Zahlen*. Friedrich Vieweg und Sohn, Braunschweig, fourth edition, 1891.